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Quantum Teleportation and Quantum Dense Coding in a Finite-Dimensional Hilbert Space

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Irreducible unitary representations of finite group and compact group describe quantum-state transformation (quantum coding) and quantum measurement (quantum decoding). The quantum teleportation and the quantum dense coding in a finite-dimensional Hilbert space are formulated in terms of an irreducible unitary representation of group. The description based on the group representation makes clear the similarity and difference between the quantum teleportation and the quantum dense coding.

KEY WORDS: quantum communication; irreducible unitary representation; quantum measurement; quantum teleportation; quantum dense coding.

1. INTRODUCTION

Quantum information has recently attracted considerable attention in quantum physics and information science and technology, which not only gives deeper insight in the principles of quantum mechanics, but also provides remarkable information processing methods such as quantum computing, quantum communication, and quantum cryptography (Hirota *et al.*, 1997; Kumar *et al.*, 2000; Tombesi and Hirota, 2001). In quantum information processing, a unitary operation is one of the most important quantum operations which transform one quantum state into another. Quantum operations include encoding some information into a quantum state and measurement performed on a quantum state. Quantum decoding that extracts information from a quantum state is nothing but a quantum measurement process which is described by positive operator-valued measure (Helstrom, 1976; Holevo, 1982). Irreducible unitary representations of finite group and compact group play a very important role in quantum coding and decoding. Unitary operators belonging to the irreducible representation not only describe a transformation of a quantum state, but also generate positive operator-valued measure of quantum measurement. Therefore general theories of the quantum teleportation and the

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quantum dense coding in a finite-dimensional Hilbert space can be formulated in terms of an irreducible representation of group.

In Section 2, quantum coding and quantum decoding based on an irreducible unitary representation (or projective representation) of group are explained. There, the standard protocol in a quantum communication system is introduced. In Sections 3 and 4, the quantum teleportation and the quantum dense coding with the standard protocol are formulated. In Section 5, simple examples of the general results are considered. In Section 6, the concluding remarks are given.

2. QUANTUM CODING AND QUANTUM DECODING

Quantum coding and quantum encoding are described by an irreducible unitary representation of group. Suppose an irreducible unitary representation (or an irreducible projective representation) $\mathcal{G} = \{\hat{U}(g) \mid g \in G\}$ of a finite group G, where a unitary operator $\hat{U}$ belonging to the representation is defined on an N-dimensional Hilbert space $\mathcal{H}$. Although a finite group is considered in this paper, the most of the results are valid for a compact group. Then, for any operator $\hat{X}$ defined on the Hilbert space $\mathcal{H}$, Schur's lemma yields the equality (Robinson, 1980)

$$\frac{N}{|G|}\sum_{g\in G}\hat{U}(g)\hat{X}\hat{U}^{\dagger}(g) = (\mathrm{Tr}\,\hat{X})\hat{1}, \tag{1}$$

where $|G|$ is the cardinality of G and $\hat{1}$ is the identity operator defined on the Hilbert space $\mathcal{H}$. Quantum measurement, the outcome of which belongs to the group G, is described by a positive operator-valued measure (POM) $\mathcal{X}_G = \{\hat{X}(g) \mid g \in G\}$ which satisfies (Helstrom, 1976; Holevo, 1982)

$$\sum_{g\in G}\hat{X}(g) = \hat{1}, \qquad \hat{X}(g) \geq 0. \tag{2}$$

When we perform the quantum measurement $\mathcal{X}_G$ on a system prepared in a quantum state $\hat{\rho}$, the measurement outcome $g \in G$ is obtained with probability

$$P(g) = \mathrm{Tr}[\hat{X}(g)\hat{\rho}], \tag{3}$$

which is normalized as $\Sigma_{g\in G}P(g) = 1$. For an arbitrary density operator $\hat{\sigma}(\hat{\sigma} \geq 0$, $\mathrm{Tr}\,\hat{\sigma} = 1)$ defined on the Hilbert space $\mathcal{H}$, we define an operator $\hat{X}_{\sigma}(g)$ as

$$\hat{X}_{\sigma}(g) = \frac{N}{|G|}\hat{U}(g)\hat{\sigma}\hat{U}^{\dagger}(g). \tag{4}$$

Then, from Schur's lemma (1), the set $\{\hat{X}_{\sigma}(g) \mid g \in G\}$ becomes a POM and describes some quantum measurement. The map $\hat{\rho} \to g$ from a quantum state to a group element determined by the POM $\{\hat{X}_{\sigma}(g) \mid g \in G\}$ is quantum decoding. Using the quantum decoding, we can extract some information from the quantum state $\hat{\rho}$.

Quantum coding is a process that encodes some information on some quantum state $\hat{\rho}$. For example, when we have the irreducible unitary representation $\mathcal{G}$, we can encode information represented by the element $g \in G$ on the quantum state $\hat{\rho}$ by the following unitary transformation:

$$\hat{\rho} \to \hat{\rho}(g) = \hat{U}(g)\hat{\rho}\hat{U}^{\dagger}(g). \tag{5}$$

In a quantum communication system, a sender can encode some information by Eq. (5) on a signal quantum state and a receiver can extract the information from the encoded quantum state by Eq. (4). The conditional probability, called the channel matrix, of such a quantum communication channel is given by

$$\begin{aligned} P(g' \mid g) &= \mathrm{Tr}[\hat{X}_{\sigma}(g')\hat{\rho}(g)] \\ &= \frac{N}{|G|}\mathrm{Tr}[\hat{U}(g^{-1}g')\hat{\sigma}\hat{U}^{\dagger}(g^{-1}g')\hat{\rho}], \end{aligned} \tag{6}$$

When the set G is an additive group, the conditional probability $P(g' \mid g)$ is a function of the difference $g' - g$, that is, $P(g' \mid g) = P(g' \mid g)$, where $P(g)$ is given by

$$P(g) = \frac{N}{|G|}\mathrm{Tr}[\hat{U}(g)\hat{\sigma}\hat{U}^{\dagger}(g)\hat{\rho}], \tag{7}$$

which is equivalent to the operational phase-space probability distribution (Ban, 1997; Bužeck *et al.*, 1996). As stated above, the irreducible unitary representation of group determines both quantum coding and quantum decoding. When the quantum coding by the sender and the quantum decoding by the receiver are determined by the irreducible unitary representation of the same group, it is said that the quantum communication is subject to the standard protocol. For example, it is the standard protocol that the sender encodes information by applying the Pauli matrices and the receiver performs the Bell measurement.

We consider a simple example of the standard protocol of quantum communication. For this purpose, we denote a complete orthonormal system of the Hilbert space $\mathcal{H}$ as $\{|\psi_0\rangle, |\psi_1\rangle, \ldots, |\psi_{N-1}\rangle\}$. We introduce N^2 unitary operators $\hat{U}(j, k)$ by

$$\begin{aligned} \hat{U}(j,k) &= \sum_{l=0}^{N-1} \exp\left(\frac{2\pi}{N} jl\right) |\psi_{l \bmod N}\rangle\langle\psi_{k+l \bmod N}| \\ &= \exp(i\theta_j\hat{n})\exp(-ik\hat{\theta}), \end{aligned} \tag{8}$$

with $j, k = 0, 1, \ldots, N-1$ and $\theta_j = 2\pi j/N$. In this equation, $\hat{\theta}$ is the Pegg–Barnett phase operator and $\hat{n}$ is the number operator canonically conjugate to $\hat{\theta}$ (Barnett and Pegg, 1990; Pegg and Barnett, 1989). The set of the operators, $\{\hat{U}(j,k) \mid j, k = 0, 1, \ldots, N-1\}$, is the irreducible projective representation of the generalized Pauli group. It is easy to see that the unitary operator $\hat{U}(j,k)$ satisfies (Ban, 2002a)

$$\frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\hat{U}(j,k)\hat{X}\hat{U}^{\dagger}(j,k) = (\mathrm{Tr}\,\hat{X})\hat{1} \tag{9}$$

for any operator $\hat{X}$ defined on the Hilbert space $\mathcal{H}$ and the orthogonality relation

$$\frac{1}{N}\mathrm{Tr}\,[\hat{U}^{\dagger}(j,k)\hat{U}(l,m)] = \delta_{jl}\delta_{km}. \tag{10}$$

Furthermore, we have the relations for the unitary operators $\hat{U}(j,k)$:

$$\hat{U}(j,k)\hat{U}(l,m)\hat{U}^{\dagger}(j,k) = \hat{U}(l,m)\exp\left[-\frac{2\pi i}{N}(jm-kl)\right], \tag{11}$$

$$\hat{U}(j,k)\hat{U}(l,m) = \hat{U}(j+l,k+m)\exp\left(\frac{2\pi i}{N}kl\right), \tag{12}$$

$$\hat{U}^{\dagger}(j,k) = \hat{U}(-j,-k)\exp\left(\frac{2\pi i}{N}jk\right), \tag{13}$$

When $N=2$, the four unitary operators $\hat{U}(j,k)$ are equivalent to the identity operator $\hat{1}$ and the three Pauli operators $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$. Thus the unitary operator $\hat{U}(j,k)$ is sometimes referred to as the generalized Pauli operator.

We can encode 2 $\log_2$ bits of information on a quantum state $\hat{\rho}$ by applying one of the N^2 unitary operators $\hat{U}(j,k)$ with equal probabilities as $\hat{\rho} \to \hat{U}(j,k)\hat{\rho}\hat{U}^{\dagger}(j,k)$. A completely entangled bipartite state $|\Psi^{\mathrm{AB}}\rangle$ of $(N\times N)$-dimensional Hilbert space $\mathcal{H}^{\mathrm{A}}\otimes\mathcal{H}^{\mathrm{B}}$ is expressed as

$$|\Psi^{\mathrm{AB}}\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\left|\Psi_k^{\mathrm{A}}\right\rangle\otimes\left|\Psi_k^{\mathrm{B}}\right\rangle. \tag{14}$$

The local unitary transformation $|\Psi_{jk}^{\mathrm{AB}}\rangle$ of the bipartite state $|\Psi^{\mathrm{AB}}\rangle$ remains completely entangled, where the bipartite quantum state $|\Psi_{jk}^{\mathrm{AB}}\rangle$ is given by

$$\left|\Psi_{jk}^{\mathrm{AB}}\right\rangle = [U^{\mathrm{A}}(j,k)\otimes I^{\mathrm{B}}]|\Psi^{\mathrm{AB}}\rangle. \tag{15}$$

When $N=2$, these quantum states become equivalent to the Bell states $|\Phi_{\pm}^{\mathrm{AB}}\rangle$ and $|\Psi_{\pm}^{\mathrm{AB}}\rangle$. It is found from Eqs. (9)–(13) that the operator $\hat{X}_{jk}^{\mathrm{AB}} = |\Psi_{jk}^{\mathrm{AB}}\rangle\langle\Psi_{jk}^{\mathrm{AB}}|$ satisfies the relations

$$\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\hat{X}_{jk}^{\mathrm{AB}} = \hat{1}^{\mathrm{A}}\otimes\hat{1}^{\mathrm{B}} \quad \text{and} \quad \hat{X}_{jk}^{\mathrm{AB}}\hat{X}_{lm}^{\mathrm{AB}} = \delta_{jl}\delta_{km}\hat{X}_{jk}^{\mathrm{AB}}. \tag{16}$$

Thus the set of the projection operators, $\{\hat{X}_{jk}^{\mathrm{AB}} \mid j,k = 0,1,\ldots,N-1\}$, describes the quantum measurement which is called the generalized Bell measurement.

3. QUANTUM TELEPORTATION

The quantum teleportation can transmit an unknown quantum state, without sending itself, by means of quantum entanglement and classical communication (Bennett *et al.*, 1993). The quantum teleportation was originally proposed for a quantum state defined on a finite-dimensional Hilbert space and generalized later (Ban, 2002b; Bowen and Bose, 2001; Braunstein *et al.*, 2000; Braunstein and Kimble, 1998). Furthermore it has been shown that an irreducible representation of group plays an important role in the ideal quantum teleportation (Braunstein *et al.*, 2000).

Suppose that Alice (a sender) and Bob (a receiver) share a bipartite quantum state $\hat{\rho}^{\mathrm{AB}}$ defined on the $(N \times N)$-dimensional Hilbert space $\mathcal{H}^{\mathrm{A}} \otimes \mathcal{H}^{\mathrm{B}}$. Furthermore Alice is provided an unknown quantum state $\hat{\rho}^{\mathrm{Q}}$ to be teleported, defined on an N-dimensional Hilbert space $\mathcal{H}^{\mathrm{Q}}$. The total quantum state of Alice and Bob is given by $\hat{\rho}_{\mathrm{in}}^{\mathrm{QAB}} = \hat{\rho}^{\mathrm{Q}} \otimes \hat{\rho}^{\mathrm{AB}}$. It is assumed here that the quantum teleportation is subject to the standard protocol explained in the previous section. Then Alice performs a quantum measurement described by an operator $\hat{X}^{\mathrm{QA}}(g)$ defined on the Hilbert space $\mathcal{H}^{\mathrm{Q}} \otimes \mathcal{H}^{\mathrm{A}}$,

$$\hat{X}^{\mathrm{QA}}(g) = |\Psi^{\mathrm{QA}}(g)\rangle\langle\Psi^{\mathrm{QA}}(g)|, \tag{17}$$

with

$$|\Psi^{\mathrm{QA}}(g)\rangle = \frac{N}{\sqrt{|G|}}(\hat{U}^{\mathrm{Q}}(g) \otimes \hat{1}^{\mathrm{A}})\Psi^{\mathrm{QA}}\rangle, \tag{18}$$

where $|\Psi^{\mathrm{QA}}\rangle$ is a completely entangled bipartite quantum state of the Hilbert space $\mathcal{H}^{\mathrm{Q}} \otimes \mathcal{H}^{\mathrm{A}}$ and the unitary operator $\hat{U}^{\mathrm{Q}}(g)$ belongs to the irreducible unitary representation $\mathcal{G}$ of the group G. It is seen from Schur's lemma (1) that the equality $\Sigma_{g\in G}\hat{X}^{\mathrm{QA}}(g) = \hat{1}^{\mathrm{Q}} \otimes \hat{1}^{\mathrm{A}}$ holds. It is important to note that the operator $\hat{X}^{\mathrm{QA}}(g)$ is not an orthogonal projector in general. After performing the quantum measurement, Alice informs Bob of the measurement outcome g by classical communication. When Bob knows the measurement outcome g, he applies the unitary operator $\hat{U}^{\mathrm{B}}(g)$ to his quantum state. Then, according to the state-reduction formula (Kraus, 1983), the quantum state $\hat{\rho}_{\mathrm{out}}^{\mathrm{B}}(g)$ that Bob finally obtains is given by

$$\hat{\rho}_{\mathrm{out}}^{\mathrm{B}}(g) = \frac{\hat{U}^{\mathrm{B}}(g)\left\{\mathrm{Tr}_{\mathrm{QA}}\left[(\hat{X}^{\mathrm{QA}}(g) \otimes \hat{1}^{\mathrm{B}})\hat{\rho}_{\mathrm{in}}^{\mathrm{QAB}}\right]\right\}\hat{U}^{\mathrm{B}\dagger}(g)}{\mathrm{Tr}_{\mathrm{QAB}}\left[(\hat{X}^{\mathrm{QA}}(g) \otimes \hat{1}^{\mathrm{B}})\hat{\rho}_{\mathrm{in}}^{\mathrm{QAB}}\right]}, \tag{19}$$

which is obtained with probability

$$P(g) = \mathrm{Tr}_{\mathrm{QAB}}\left[(\hat{X}^{\mathrm{QA}}(g) \otimes \hat{1}^{\mathrm{B}})\hat{\rho}_{\mathrm{in}}^{\mathrm{QAB}}\right]. \tag{20}$$

Therefore Bob obtains, in average, the quantum state $\hat{\rho}^{\mathrm{B}}_{\mathrm{out}}$,

$$\hat{\rho}^{\mathrm{B}}_{\mathrm{out}} \sum_{g\in G} \hat{U}^{\mathrm{B}}(g) \left\{ \mathrm{Tr}_{\mathrm{QA}} \left[(\hat{X}^{\mathrm{QA}}(g) \otimes \hat{1}^{\mathrm{B}}) \hat{\rho}^{\mathrm{QAB}}_{\mathrm{in}} \right] \right\} \hat{U}^{\mathrm{B}\dagger}(g). \tag{21}$$

In the ideal quantum teleportation, the quantum states $\hat{\rho}^{\mathrm{B}}_{\mathrm{out}}(g)$ and $\hat{\rho}^{\mathrm{B}}_{\mathrm{out}}$ are equal to the unknown quantum state that Alice was given.

In the standard protocol, substituting Eqs. (17) and (18) into Eqs. (19)–(21), we obtain

$$\hat{\rho}^{\mathrm{B}}_{\mathrm{out}}(g) = \frac{\langle \Psi^{\mathrm{QA}} | [\hat{\rho}^{\mathrm{Q}} \otimes \hat{\rho}^{\mathrm{AB}}(g)] | \Psi^{\mathrm{QA}} \rangle}{\mathrm{Tr}_{\mathrm{B}} \langle \Psi^{\mathrm{QA}} | [\hat{\rho}^{\mathrm{Q}} \otimes \hat{\rho}^{\mathrm{AB}}(g)] | \Psi^{\mathrm{QA}} \rangle}, \tag{22}$$

$$P(g) = \frac{N^2}{|G|} \mathrm{Tr}_{\mathrm{B}} \langle \Psi^{\mathrm{QA}} | [\hat{\rho}^{\mathrm{Q}} \otimes \hat{\rho}^{\mathrm{AB}}(g)] | \Psi^{\mathrm{QA}} \rangle, \tag{23}$$

$$\hat{\rho}^{\mathrm{B}}_{\mathrm{out}} = N^2 \left\langle \Psi^{\mathrm{QA}} \middle| \left(\hat{\rho}^{\mathrm{Q}} \otimes \hat{\rho}^{\mathrm{AB}}_{G} \right) \middle| \Psi^{\mathrm{QA}} \right\rangle, \tag{24}$$

where the quantum state $\hat{\rho}^{\mathrm{AB}}(g)$ is the twirling transformation of the bipartite quantum state $\hat{\rho}^{\mathrm{AB}}$ (Horodecki *et al.*, 1999),

$$\hat{\rho}^{\mathrm{AB}}(g) = [\hat{U}^{\mathrm{A}*}(g) \otimes \hat{U}^{\mathrm{B}}(g)] \hat{\rho}^{\mathrm{AB}} [\hat{U}^{\mathrm{A}*}(g) \otimes \hat{U}^{\mathrm{B}}(g)]^{\dagger}, \tag{25}$$

and the quantum state $\hat{\rho}^{\mathrm{AB}}_{G}$ is the average of all the possible transformation,

$$\hat{\rho}^{\mathrm{AB}}_{G} = \frac{1}{|G|} \sum_{g\in G} \hat{\rho}^{\mathrm{AB}}(g). \tag{26}$$

In deriving these equations, we have used the fact that the following relation holds for the completely entangled state $|\Psi^{\mathrm{QA}}\rangle$:

$$(\hat{V}^{\mathrm{Q}} \otimes \hat{1}^{\mathrm{QA}}) | \Psi^{\mathrm{QA}} \rangle = (\hat{1}^{\mathrm{Q}} \otimes \hat{V}^{\mathrm{AT}}) | \Psi^{\mathrm{QA}} \rangle, \tag{27}$$

where the symbol T stands for the transposition of the operator.

In particular, if the unknown quantum state $\hat{\rho}^{\mathrm{Q}}$ is pure, that is, $\hat{\rho}^{\mathrm{Q}} = |\psi^{\mathrm{Q}}\rangle \langle \Psi^{\mathrm{Q}}|$, Eqs. (22)–(24) are simplified as

$$\hat{\rho}^{\mathrm{B}}_{\mathrm{out}}(g) = \frac{\langle \Psi^{\mathrm{A}*} | \hat{\rho}^{\mathrm{AB}}(g) | \Psi^{\mathrm{A}*} \rangle}{\mathrm{Tr}_{\mathrm{B}} \langle \Psi^{\mathrm{A}*} | \hat{\rho}^{\mathrm{AB}}(g) | \Psi^{\mathrm{A}*} \rangle}, \tag{28}$$

$$P(g) = \frac{N}{|G|} \mathrm{Tr}_{\mathrm{B}} \langle \Psi^{\mathrm{A}*} | \hat{\rho}^{\mathrm{AB}}(g) | \Psi^{\mathrm{A}*} \rangle, \tag{29}$$

$$\hat{\rho}^{\mathrm{B}}_{\mathrm{out}} = N \left\langle \Psi^{\mathrm{A}*} \middle| \hat{\rho}^{\mathrm{AB}}_{G} \middle| \Psi^{\mathrm{A}*} \right\rangle. \tag{30}$$

Here, when the state vector $|\Psi\rangle$ is expanded as $|\Psi\rangle = \Sigma_{k=0}^{N-1} a_k |\Psi_k\rangle$, the state vector $|\Psi^*\rangle$ is defined by $|\Psi^*\rangle = \Sigma_{k=0}^{N-1} a_k^* |\Psi_k\rangle$, the set $\{|\Psi_k\rangle | k = 0, 1, \ldots, N-1\}$

being a complete orthonormal system of an N-dimensional Hilbert space. Furthermore we used the fact that

$$\sum_{k=0}^{N-1} \left\langle \Psi_k^{\mathrm{Q}} \middle| \Psi^{\mathrm{Q}} \right\rangle \left| \Psi_k^{\mathrm{A}} \right\rangle = |\Psi^{\mathrm{A}}\rangle. \tag{31}$$

In this case, the fidelity $\mathcal{F}(g)$ and the average value of the fidelity $\bar{\mathcal{F}}$ are given by

$$\mathcal{F}(g) = \frac{\langle \Psi^{\mathrm{A}*}, \Psi^{\mathrm{B}} | \hat{\rho}^{\mathrm{AB}}(g) | \Psi^{\mathrm{A}*}, \Psi^{\mathrm{B}} \rangle}{\mathrm{Tr}_{\mathrm{B}} \langle \Psi^{\mathrm{A}*} | \hat{\rho}^{\mathrm{AB}}(g) | \Psi^{\mathrm{A}*} \rangle}, \tag{32}$$

$$\bar{\mathcal{F}} = \left\langle \Psi^{\mathrm{A}*}, \Psi^{\mathrm{B}} \middle| \hat{\rho}_G^{\mathrm{AB}} \middle| \Psi^{\mathrm{A}*}, \Psi^{\mathrm{B}} \right\rangle, \tag{33}$$

where $|\Psi^{\mathrm{A}*}, \Psi^{\mathrm{B}}\rangle = |\Psi^{\mathrm{A}*}\rangle \otimes |\Psi^{\mathrm{B}}\rangle$.

Suppose that the bipartite quantum state $\hat{\rho}^{\mathrm{AB}}$ shared by Alice and Bob is the generalized Werner state given by

$$\hat{\rho}_{\mathrm{W}}^{\mathrm{AB}} = F |\Psi^{\mathrm{AB}}\rangle\langle\Psi^{\mathrm{AB}}| + \frac{1-F}{N^2-1} (\hat{1}^{\mathrm{A}} \otimes \hat{1}^{\mathrm{B}} - |\Psi^{\mathrm{AB}}\rangle\langle\Psi^{\mathrm{AB}}|), \tag{34}$$

where the singlet fraction F satisfies $0 \le F \le 1$. In this case, Eqs. (22)–(24) are calculated to be

$$\hat{\rho}_{\mathrm{out}}^{\mathrm{B}} = \hat{\rho}_{\mathrm{out}}^{\mathrm{B}}(g) = \frac{N^2 F - 1}{N^2 - 1} \hat{\rho}^{\mathrm{B}} + \frac{N^2(1-F)}{N^2-1} (\hat{1}^{\mathrm{B}}/N), \tag{35}$$

and $P(g) = 1/|G|$, where the density operator $\hat{\rho}^{\mathrm{B}}$ represents the unknown quantum state to be teleported, defined in the Hilbert space $\mathcal{H}^{\mathrm{B}}$. In this case, Alice and Bob cannot obtain any information about the unknown quantum state $\hat{\rho}^{\mathrm{Q}}$. The result means that even if the operator $\hat{\underline{X}}^{\mathrm{QA}}(g)$ is not an orthogonal projector, the perfect quantum teleportation ($\mathcal{F}(g) = \bar{\mathcal{F}} = 1$) is possible when Alice and Bob share the completely entangled state ($F = 1$) (Braunstein *et al.*, 2000). This result is quite different from that obtained for the quantum dense coding (see the next section). Let the eigenvalues of the unknown quantum state be $\lambda_0, \lambda_1, \ldots \lambda_{N-1}$. Then the fidelity is calculated to be

$$\begin{aligned} \bar{\mathcal{F}} = \mathcal{F}(g) &= \left(\mathrm{Tr} \sqrt{\sqrt{\hat{\rho}^{\mathrm{B}}} \hat{\rho}_{\mathrm{out}}^{\mathrm{B}} \sqrt{\hat{\rho}^{\mathrm{B}}}} \right)^2 \\ &= \left(\sum_{k=0}^{N-1} \sqrt{\frac{N^2 F - 1}{N^2 - 1} \lambda_k^2 + \frac{N(1-F)}{N^2-1} \lambda_k} \right)^2 \\ &\le \frac{NF+1}{N+1}, \end{aligned} \tag{36}$$

where the equality holds for $F = 1$ or $(\hat{\rho}^{\mathrm{B}})^2 = \hat{\rho}^{\mathrm{B}}$.

We next assume that the unitary operator $\hat{U}(g)$ is given by Eq. (8). In this case, we have $g = (j, k)$ and $|G| = N^2$. Since the unitary operator $\hat{U}(g)$ satisfies Eqs. (9)–(13), the operator $\hat{X}^{\mathrm{QA}}(g)$ becomes an orthogonal projector and describes the generalized Bell measurement. Furthermore, the set $\{|\Psi^{\mathrm{AB}}(j, k)\rangle = |\Psi^{\mathrm{AB}}_{jk}\rangle \mid j, k = 0, 1, \ldots N-1\}$ is an complete orthonormal system of the Hilbert space $\mathcal{H}^{\mathrm{A}} \otimes \mathcal{H}^{\mathrm{B}}$. Using these properties, after some calculation, we obtain the relation

$$\begin{aligned}&\langle \Psi^{\mathrm{QA}}|[\hat{\rho}^{\mathrm{Q}} \otimes \hat{\rho}^{\mathrm{AB}}(j, k)]|\Psi^{\mathrm{QA}}\rangle \\ &\quad = \frac{1}{N^2} \sum_{l=0}^{N-1} \sum_{l'=0}^{N-1} \sum_{n=0}^{N-1} \sum_{n'=0}^{N-1} \hat{U}^{\mathrm{B}}(l, -l')\hat{\rho}^{\mathrm{B}}\hat{U}^{\mathrm{B}\dagger}(n, -n')\big\langle \Psi^{\mathrm{AB}}_{ll'}|\hat{\rho}^{\mathrm{AB}}|\Psi^{\mathrm{AB}}_{nn'}\big\rangle \\ &\qquad \times e^{-i(2\pi/N)(ll'-nn')+i(2\pi/N)k(l-n)+i(2\pi/N)j(l'-n')},\end{aligned} \tag{37}$$

where we have used the fact that

$$\sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \big|\Psi^{\mathrm{B}}_j\big\rangle\big\langle \Psi^{\mathrm{Q}}_k|\hat{\rho}^{\mathrm{Q}}|\Psi^{\mathrm{Q}}_k\big\rangle\big\langle \Psi^{\mathrm{B}}_k\big| = \hat{\rho}^{\mathrm{B}}. \tag{38}$$

Substituting Eq. (37) into Eqs. (22)–(24), we can obtain all the quantities that completely characterize the quantum teleportation. In particular, the averaged quantum state $\hat{\rho}^{\mathrm{B}}_{\mathrm{out}}$ is given by

$$\hat{\rho}^{\mathrm{B}}_{\mathrm{out}} = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \mathcal{P}(j, k)\hat{U}^{\mathrm{B}}(j, -k)\hat{\rho}^{\mathrm{B}}\hat{U}^{\mathrm{B}\dagger}(j, -k), \tag{39}$$

with

$$\mathcal{P}(j, k) = \big\langle \Psi^{\mathrm{AB}}_{jk}|\hat{\rho}^{\mathrm{AB}}|\Psi^{\mathrm{AB}}_{jk}\big\rangle. \tag{40}$$

This result means that the quantum teleportation is equivalent, in average, to the generalized depolarizing channel (Bowen and Bose, 2001).

4. QUANTUM DENSE CODING

The quantum dense coding transmits classical information by means of quantum entanglement and quantum communication (Ban, 2002a; Bennet and Wiesner, 1992; Bowen, 2001). The amount of classical information transmitted via the quantum dense coding is greater than that transmitted without using quantum entanglement. In ideal cases, the information capacity of the quantum dense coding channel is twice as great as the capacity of the quantum channel without quantum entanglement. For example, although the upper bound on the capacity of a qubit channel is one bit, the quantum dense coding of a qubit can transmit two bits of information.

Suppose that Alice and Bob share a bipartite quantum state $\hat{\rho}^{\mathrm{AB}}$ defined on the $(N \times N)$-dimensional Hilbert space $\mathcal{H}^{\mathrm{A}} \otimes \mathcal{H}^{\mathrm{B}}$. In the quantum dense coding which obeys the standard protocol, Alice encodes classical information on her quantum state by applying the unitary operator $\hat{U}^{\mathrm{A}}(g)$ which belongs to the irreducible unitary representation $\mathcal{G}$ of the group G, and she sends the encoded quantum state to Bob through a noiseless quantum channel. Then Bob obtains the encoded bipartite quantum state

$$\hat{\rho}^{\mathrm{AB}}(g) = (\hat{U}^{\mathrm{A}}(g) \otimes \hat{1}^{\mathrm{B}})\hat{\rho}^{\mathrm{AB}}(\hat{U}^{\mathrm{A}\dagger}(g) \otimes \hat{1}^{\mathrm{B}}). \tag{41}$$

To extract the information that Alice encoded, Bob performs the quantum measurement described by the operator $\hat{X}^{\mathrm{AB}}(g)$ defined by Eqs. (17) and (18). The conditional probability (the channel matrix of the quantum dense coding channel) $P(g' \mid g)$ that Bob obtains the measurement outcome g' when Alice encoded the information g is given by

$$P(g' \mid g) = \mathrm{Tr}_{\mathrm{AB}}[\hat{X}^{\mathrm{AB}}(g')\hat{\rho}^{\mathrm{AB}}(g)]. \tag{42}$$

Substituting Eqs. (17), (18), and (41) into this equation, the conditional probability $P(g' \mid g)$ can be expressed as

$$P(g' \mid g) = \frac{N^2}{|G|}\langle \Psi^{\mathrm{AB}}|\hat{\rho}^{\mathrm{AB}}(g, g')|\Psi^{\mathrm{AB}}\rangle, \tag{43}$$

with

$$\hat{\rho}^{\mathrm{AB}}(g, g') = [\hat{U}^{\mathrm{A}}(g) \otimes \hat{U}^{\mathrm{B}*}(g')]\hat{\rho}^{\mathrm{AB}}[\hat{U}^{\mathrm{A}}(g) \otimes \hat{U}^{\mathrm{B}*}(g')]^{\dagger}. \tag{44}$$

When Alice encodes the information g on her quantum state with probability $\pi(g)$, the mutual information $I(B:A)$ of the quantum dense coding channel becomes

$$I(B:A) = \sum_{g \in G}\sum_{g' \in G} P(g' \mid g)\pi(g) \log\left[\frac{P(g' \mid g)}{\sum_{g'' \in G} P(g' \mid g'')\pi(g'')}\right]. \tag{45}$$

It is easy to see from Schur's lemma (1) that if the prior probabilities are equal ($\pi(g) = 1/|G|$), the output probabilities $P_{\mathrm{out}}(g')$ are also equal:

$$P_{\mathrm{out}}(g') = \sum_{g \in G} P(g' \mid g)\frac{1}{|G|} = \frac{1}{|G|}. \tag{46}$$

In this case, the mutual information $I(B:A)$ becomes

$$I(B:A) = \log|G| + \sum_{g \in G}\sum_{g' \in G} P(g' \mid g) \log P(g' \mid g). \tag{47}$$

To obtain the maximum value of the mutual information, the equality $P(g' \mid g) = \delta_{gg'}$ must be fulfilled. To investigate this condition, we assume here that Alice and Bob share the completely entangled bipartite quantum state $\hat{\rho}^{\mathrm{AB}} = |\Psi^{\mathrm{AB}}\rangle\langle\Psi^{\mathrm{AB}}|$. Then, the conditional probability $P(g' \mid g)$ is given by

$$P(g' \mid g) = |\langle\Psi^{\mathrm{AB}}(g')|\Psi^{\mathrm{AB}}(g)\rangle|^2, \tag{48}$$

where $|\Psi^{\mathrm{AB}}(g)\rangle$ is given by Eq. (18). Thus to establish the equality $P(g' \mid g) = \delta_{gg'}$, the state vector $|\Psi^{\mathrm{AB}}(g)\rangle$ must be orthogonal. To satisfy the condition, it is enough that the unitary operator $\hat{U}(g)$ is chosen to be $\hat{U}(j, k)$, which is given by Eq. (8). In this case, we obtain $I(B : A) = 2 \log N$, which is twice as great as the information transmitted without the quantum entanglement. Here, it is important to note that although it is sufficient for performing the perfect quantum teleportation that Alice and Bob share completely entangled bipartite state, the error-free quantum dense coding which yields $I(B : A) = 2 \log N$ further requires that the bipartite quantum state $|\Psi^{\mathrm{AB}}(g)\rangle$ be orthogonal.

The quantum information theory tells us that the classical information capacity C of the quantum dense coding channel is given by (Holevo, 1998; Schumacher and Westmoreland, 1997)

$$C = \max_{\pi(g)} \left[S\left(\sum_{g\in G} \pi(g)\hat{\rho}^{\mathrm{AB}}(g) \right) - \sum_{g\in G} \pi(g) S(\hat{\rho}^{\mathrm{AB}}(g)) \right], \tag{49}$$

where $S(\hat{\rho})$ is the von Neumann entropy of the quantum state $\hat{\rho}$,

$$S(\hat{\rho}) = -\mathrm{Tr}[\hat{\rho} \log \hat{\rho}]. \tag{50}$$

It has been shown (Ban, 2002a; Bowen, 2001) that the maximum value of the capacity is attained when Alice applies the unitary operator $\hat{U}(j, k)$ with equal probabilities ($\pi(j, k) = 1/N^2$) and Bob performs the generalized Bell measurement $\hat{X}^{\mathrm{AB}}_{jk}$. In this case, the capacity of the quantum dense coding channel becomes

$$C = \log N + S(\hat{\rho}^{\mathrm{B}}) - S(\hat{\rho}^{\mathrm{AB}}), \tag{51}$$

where $\hat{\rho}^{\mathrm{B}} = \mathrm{Tr}_{\mathrm{A}}\, \hat{\rho}^{\mathrm{AB}}$. When $\hat{\rho}^{\mathrm{AB}}$ is the completely entangled state, we obtain $C = 2 \log N$ since $\hat{\rho}^{\mathrm{B}} = \hat{1}^{\mathrm{B}}/N$. This result means that the maximum value of the capacity is attained by means of the completely entangled bipartite state while the maximum value of the mutual information further needs the orthogonality of the completely entangled state.

5. SIMPLE EXAMPLE

In this section, we consider the quantum teleportation and the quantum dense coding of a qubit, where Alice and Bob share one of the four Bell states,

$$|\Psi_{\pm}^{\mathrm{AB}}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, \qquad |\Psi_{\pm}^{\mathrm{AB}}\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}. \tag{52}$$

When the set of the Pauli matrices $\{\hat{1}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$ is used as the irreducible projective representation of the group, an arbitrary qubit state can be sent perfectly by means of the quantum teleportation and two bits of classical information can be sent via the quantum dense coding (Bennet and Wiesner, 1992; Bennett *et al.*, 1993).

Suppose that Alice and Bob use the irreducible unitary representation of the point group $\mathbf{D}_3$, which is the symmetric group of a regular triangle, instead of the Pauli group, where the representation consists of the following 2×2 unitary matrices:

$$\hat{U}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \hat{U}_2 = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \tag{53}$$

$$\hat{U}_3 = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}, \qquad \hat{U}_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{54}$$

$$\hat{U}_5 = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}, \qquad \hat{U}_6 = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}. \tag{55}$$

Even in this case, since Alice and Bob share the completely entangled state, an arbitrary qubit state can be transmitted with unity of the fidelity by the quantum teleportation, where $2\ \log_2 3$ bits of classical information should be sent from Alice to Bob. To investigate the quantum dense coding, we assume that Alice and Bob share the Bell state $|\Phi_{+}^{\mathrm{AB}}\rangle$. Then the state vectors $|\Psi_{j}^{\mathrm{AB}}\rangle = (\hat{U}_j^{\mathrm{A}} \otimes \hat{1}^{B})|\Phi_{+}^{\mathrm{AB}}\rangle (j = 1, 2, \ldots, 6)$ become

$$\left|\Psi_1^{\mathrm{AB}}\right\rangle = |\Phi_{+}^{\mathrm{AB}}\rangle, \qquad \left|\Psi_2^{\mathrm{AB}}\right\rangle = \frac{1}{2}|\Phi_{+}^{\mathrm{AB}}\rangle + \frac{\sqrt{3}}{2}|\Psi_{-}^{\mathrm{AB}}\rangle, \tag{56}$$

$$\left|\Psi_3^{\mathrm{AB}}\right\rangle = \frac{1}{2}|\Phi_{+}^{\mathrm{AB}}\rangle - \frac{\sqrt{3}}{2}|\Psi_{-}^{\mathrm{AB}}\rangle, \tag{57}$$

$$\left|\Psi_4^{\mathrm{AB}}\right\rangle = |\Phi_{-}^{\mathrm{AB}}\rangle, \qquad \left|\Psi_5^{\mathrm{AB}}\right\rangle = \frac{1}{2}|\Phi_{-}^{\mathrm{AB}}\rangle + \frac{\sqrt{3}}{2}|\Psi_{+}^{\mathrm{AB}}\rangle, \tag{58}$$

$$\left|\Psi_6^{\mathrm{AB}}\right\rangle = \frac{1}{2}|\Phi_{-}^{\mathrm{AB}}\rangle - \frac{\sqrt{3}}{2}|\Psi_{+}^{\mathrm{AB}}\rangle. \tag{59}$$

The conditional probability $P(j \mid k)$ of the quantum dense coding channel is given by

$$P(j \mid k) = \begin{pmatrix} 2/3 & 1/6 & 1/6 & 0 & 0 & 0 \\ 1/6 & 2/3 & 1/6 & 0 & 0 & 0 \\ 1/6 & 1/6 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2/3 & 1/6 & 1/6 \\ 0 & 0 & 0 & 1/6 & 2/3 & 1/6 \\ 0 & 0 & 0 & 1/6 & 1/6 & 2/3 \end{pmatrix}. \tag{60}$$

When Alice applies the unitary operator $\hat{U}_j$ with equal probabilities ($\pi_j = 1/6$), the information transmitted from Alice to Bob is calculated to be $I(B:A) = 4/3$ (bits). Note that when Alice use the Pauli matrices to encode the information and Bob performs the Bell measurement, $I(B:A) = 2$ (bits) is obtained.

6. CONCLUDING REMARKS

We have formulated the quantum teleportation and the quantum dense coding in a finite-dimensional Hilbert space in terms of the irreducible unitary representation of group. Such a formulation makes clear the similarity and difference between the quantum teleportation and the quantum dense coding. Although we have confined ourselves to a finite group and its representation in this paper, the results can be generalized for an infinite group that satisfies the compactness. In this case, the summation $(N/|G)\Sigma_{g\in G} F(g)$ is replaced with the integral $\int_{g\in G} d\mu(g)\, F(g)$, where $d\mu\,(g)$ is the invariant Haar measure (Robinson, 1980).

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